



**University of
Zurich**^{UZH}

**Zurich Open Repository and
Archive**

University of Zurich
University Library
Strickhofstrasse 39
CH-8057 Zurich
www.zora.uzh.ch

Year: 2012

An algebraic approach for decoding spread codes

Gorla, E ; Manganiello, F ; Rosenthal, J

Abstract: In this paper we study spread codes: a family of constant-dimension codes for random linear network coding. In other words, the codewords are full-rank matrices of size $k \times n$ with entries in a finite field F_q . Spread codes are a family of optimal codes with maximal minimum distance. We give a minimum-distance decoding algorithm which requires $O((n-k)k^3)$ operations over an extension field F_{q^k} . Our algorithm is more efficient than the previous ones in the literature, when the dimension k of the codewords is small with respect to n . The decoding algorithm takes advantage of the algebraic structure of the code, and it uses original results on minors of a matrix and on the factorization of polynomials over finite fields.

DOI: <https://doi.org/10.3934/amc.2012.6.443>

Posted at the Zurich Open Repository and Archive, University of Zurich

ZORA URL: <https://doi.org/10.5167/uzh-70728>

Journal Article

Published Version

Originally published at:

Gorla, E; Manganiello, F; Rosenthal, J (2012). An algebraic approach for decoding spread codes. *Advances in Mathematics of Communication*, 6(4):443-466.

DOI: <https://doi.org/10.3934/amc.2012.6.443>

AN ALGEBRAIC APPROACH FOR DECODING SPREAD CODES

ELISA GORLA

Institut de Mathématiques
Université de Neuchâtel
2000 Neuchâtel, Switzerland

FELICE MANGANIELLO

Department of Electrical and Computer Engineering
University of Toronto
Toronto, Ontario, M5S 3G4, Canada

JOACHIM ROSENTHAL

Institut für Mathematik
Universität Zürich
8057 Zürich, Switzerland

(Communicated by Marcus Greferath)

ABSTRACT. In this paper we study spread codes: a family of constant-dimension codes for random linear network coding. In other words, the codewords are full-rank matrices of size $k \times n$ with entries in a finite field \mathbb{F}_q . Spread codes are a family of optimal codes with maximal minimum distance. We give a minimum-distance decoding algorithm which requires $\mathcal{O}((n-k)k^3)$ operations over an extension field \mathbb{F}_{q^k} . Our algorithm is more efficient than the previous ones in the literature, when the dimension k of the codewords is small with respect to n . The decoding algorithm takes advantage of the algebraic structure of the code, and it uses original results on minors of a matrix and on the factorization of polynomials over finite fields.

1. INTRODUCTION

Network coding is a branch of coding theory that arose in 2000 in the work by Ahlswede, Cai, Li and Yeung [1]. While classical coding theory focuses on point-to-point communication, network coding focuses on multicast communication, i.e., a source communicating with a set of sinks. The source transmits messages to the sinks over a network, which is modeled as a directed multigraph. Some examples of multicast communication come from Internet protocol applications of streaming media, digital television, and peer-to-peer networking.

The goal in multicast communication is achieving maximal information rate. Informally, this corresponds to maximizing the amount of messages per transmission,

2000 *Mathematics Subject Classification*: 11T71.

Key words and phrases: Random linear network coding, spread codes, decoding algorithm.

The first author was supported by the Swiss National Science Foundation under grant no. 123393.

The second author was partially supported by the Swiss National Science Foundation under grants no. 126948 and no. 135934.

The third author was partially supported by the Swiss National Science Foundation under grant no. 126948.

i.e., per single use of the network. Li, Cai and Yeung in [9] prove that maximal information rate can be achieved in multicast communication using linear network coding, provided that the size of the base field is large enough.

The algebraic aspect of network coding emerged with the work by Kötter and Kschischang [8]. The authors introduced a new setting for random linear network coding: Given the linearity of the combinations, the authors suggest to employ subspaces of a given vector space as codewords. Indeed, subspaces are invariant under taking linear combinations of their elements. Let $\mathcal{P}(\mathbb{F}_q^n)$ be the set of all \mathbb{F}_q -linear subspaces of \mathbb{F}_q^n . They show that $\mathcal{P}(\mathbb{F}_q^n)$ is a metric space, with distance

$$d(\mathcal{U}, \mathcal{V}) = \dim(\mathcal{U} + \mathcal{V}) - \dim(\mathcal{U} \cap \mathcal{V}) \text{ for all } \mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathbb{F}_q^n).$$

Kötter and Kschischang define network codes to be subsets of $\mathcal{P}(\mathbb{F}_q^n)$. In particular, they define constant-dimension codes as subsets, whose elements have all the same dimension. Notions of errors and erasures compatible with the new transmission model are introduced in [8]. In addition, upper and lower bounds for the cardinality of network codes are established in [8, 3].

We review here some of the constructions of constant-dimension codes present in the literature. The first one is introduced by Kötter and Kschischang in [8]. The construction uses evaluation of linearized polynomials over a subspace. The codes that one obtains are called Reed-Solomon-like codes, because of the similarities with Reed-Solomon codes in classical coding theory. Due to their connection with the rank-metric codes introduced in [4], these codes are also called lifted rank-metric codes. Kötter and Kschischang devise a list-1 minimum-distance decoding algorithm for their codes. Spread codes, which are the subject of this paper, were first introduced by the authors in [13]. Spread codes contain the codes with maximal minimum distance in [8]. Another family of network codes, based on q -analogs of designs, appears in [7]. Aided by computer search, the authors find constant-dimension codes based on designs with big cardinality. Another family of codes is constructed in [2]. The construction builds on that of Reed-Solomon-like codes, and the codes that the authors obtain contain them. The construction is also based on binary constant-weight codes, Ferrer diagrams, and rank-metric codes. The proposed decoding algorithm operates on two levels: First one decodes a constant-weight code, then one applies a decoding algorithm for rank-metric codes. In [16] Skachek introduces a family of codes, that is a sub-family of the one in [2]. In [12] the authors introduce another family of codes, which they obtain by evaluating pairs of linearized polynomials. The codes obtained can be decoded via a list decoding algorithm, which is introduced in the same work.

This work focuses on *spread codes* which are a family of constant-dimension codes first introduced in [13]. Spreads of \mathbb{F}_q^n are a collection of subspaces of \mathbb{F}_q^n , all of the same dimension, which partition the ambient space. Such a family of subspaces of \mathbb{F}_q^n exists if and only if the dimension of the subspaces divides n . The construction of spread codes is based on the \mathbb{F}_q -algebra $\mathbb{F}_q[P]$ where $P \in GL_k(\mathbb{F}_q)$ is the companion matrix of a monic irreducible polynomial of degree k . Concretely, we define spread codes as

$$\mathcal{S}_r = \{\text{rowsp}(A_1 \ \cdots \ A_r) \in \mathfrak{G}_{\mathbb{F}_q}(k, n) \mid A_i \in \mathbb{F}_q[P] \text{ for all } i \in \{1, \dots, r\}\}$$

where $\mathfrak{G}_{\mathbb{F}_q}(k, n)$ is the Grassmannian of all subspaces of \mathbb{F}_q^n of dimension k .

Since spreads partition the ambient space, spread codes are optimal. More precisely, they have maximum possible minimum distance $2k$, and the largest possible number of codewords for a code with minimum distance $2k$. Indeed, they achieve

the anticode bound from [3]. This family is closely related to the family of Reed-Solomon-like codes introduced in [8]. We discuss the relation in detail in Section 2.2. In Lemma 17, we show how to extend to spread codes the existing decoding algorithms for Reed-Solomon-like codes and rank-metric codes.

The structure of the spreads that we use in our construction helps us devise a minimum-distance decoding algorithm, which can correct up to half the minimum distance of S_r . In Lemma 28 we reduce the decoding algorithm for a spread code S_r to at most $r - 1$ instances of the decoding algorithm for the special case $r = 2$. Therefore, we focus on the design of a decoding algorithm for the spread code

$$S = S_2 = \{\text{rowsp}(A_1 \ A_2) \in \mathfrak{G}_{\mathbb{F}_q}(k, 2k) \mid A_1, A_2 \in \mathbb{F}_q[P]\}.$$

The paper is structured as follows. In Section 2 we give the construction of spread codes, discuss their main properties. In Subsection 2.1 we introduce the main notations. In Subsection 2.2 we discuss the relation between spread codes and Reed-Solomon-like codes, which is given explicitly in Proposition 15. Proposition 18 shows how to apply a minimum-distance decoding algorithm for Reed-Solomon-like codes to spread codes, and estimates the complexity of decoding a spread code using such an algorithm.

The main results of the paper are contained in Section 3. In Subsection 3.1 we prove some results on matrices, which will be needed for our decoding algorithm. Our main result is a new minimum-distance decoding algorithm for spread codes, which is given in pseudocode as Algorithm 2. The decoding algorithm is based on Theorem 34, where we explicitly construct the output of the decoder. Our algorithm can be made more efficient when the first k columns of the received word are linearly independent. Proposition 35 and Corollary 36 contain the theoretical results behind this simplification, and the algorithm in pseudocode is given in Algorithm 3. Finally, in Section 4 we compute the complexity of our algorithm. Using the results from Subsection 2.2, we compare it with the complexity of the algorithms in the literature. It turns out that our algorithm is more efficient than all the known ones, provided that $k \ll n$.

2. PRELIMINARIES AND NOTATIONS

Definition 1 ([6, Section 4.1]). A subset $S \subset \mathfrak{G}_{\mathbb{F}_q}(k, n)$ is a spread if it satisfies

- $\mathcal{U} \cap \mathcal{V} = \{0\}$ for all $\mathcal{U}, \mathcal{V} \in S$ distinct, and
- $\mathbb{F}_q^n = \bigcup_{\mathcal{U} \in S} \mathcal{U}$.

Theorem 2 ([6, Theorem 4.1]). A spread exists if and only if $k \mid n$.

In [13] we give a construction of spreads suitable for use in Random Linear Network Coding (RLNC). Our construction is based on companion matrices.

Definition 3. Let \mathbb{F}_q be a finite field and $p = \sum_{i=0}^k p_i x^i \in \mathbb{F}_q[x]$ a monic polynomial. The companion matrix of p

$$P = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ -p_0 & -p_1 & -p_2 & \cdots & -p_{k-1} \end{pmatrix} \in \mathbb{F}_q^{k \times k}.$$

Let $n = rk$ with $r > 1$, $p \in \mathbb{F}_q[x]$ a monic irreducible polynomial of degree k and $P \in \mathbb{F}_q^{k \times k}$ its companion matrix.

Lemma 4. *The \mathbb{F}_q -algebra $\mathbb{F}_q[P]$ is a finite field, i.e., $\mathbb{F}_q[P] \cong \mathbb{F}_{q^k}$.*

This is a well-known fact (see [10, page 64]).

Lemma 5. *Let $\varphi : \mathbb{F}_{q^k} \rightarrow \mathbb{F}_q[P]$ be a ring isomorphism. Denote by*

$$\mathbb{P}^{r-1}(\mathbb{F}_{q^k}) := (\mathbb{F}_{q^k}^r \setminus \{0\}) / \sim$$

the projective space, where \sim is the following equivalence relation

$$v \sim w \iff \exists \lambda \in \mathbb{F}_{q^k}^* \text{ such that } v = \lambda w,$$

where $v, w \in \mathbb{F}_{q^k}^r \setminus \{0\}$. Then the map

$$\begin{aligned} \tilde{\varphi} : \mathbb{P}^{r-1}(\mathbb{F}_{q^k}) &\rightarrow \mathfrak{G}_{\mathbb{F}_q}(k, n) \\ [v_1 : \dots : v_r] &\mapsto \text{rowsp}(\varphi(v_1) \quad \dots \quad \varphi(v_r)). \end{aligned}$$

is injective.

Proof. Let $v = [v_1 : \dots : v_r], w = [w_1 : \dots : w_r] \in \mathbb{P}^{r-1}(\mathbb{F}_{q^k})$. If $\tilde{\varphi}(v) = \tilde{\varphi}(w)$ there exists an $M \in GL_k(\mathbb{F}_q)$ such that

$$\begin{aligned} (\varphi(v_1) \quad \dots \quad \varphi(v_r)) &= M(\varphi(w_1) \quad \dots \quad \varphi(w_r)) \\ (1) \quad &= (M\varphi(w_1) \quad \dots \quad M\varphi(w_r)). \end{aligned}$$

Let $i_v, i_w \in \{1, \dots, r\}$ be the least indices such that $\varphi(v_{i_v}) \neq 0$ and $\varphi(w_{i_w}) \neq 0$. From (1) it follows that $i_v = i_w$. Since, without loss of generality, we can consider $v_{i_v} = w_{i_w} = 1$, it follows that $\varphi(v_{i_v}) = \varphi(w_{i_w}) = I$ and consequently $M = I$. Then, (1) becomes

$$(\varphi(v_1) \quad \dots \quad \varphi(v_r)) = (\varphi(w_1) \quad \dots \quad \varphi(w_r))$$

leading to $v = w$. □

Theorem 6 ([13, Theorem 1]). $S_r := \tilde{\varphi}(\mathbb{P}^{r-1}(\mathbb{F}_{q^k}))$ is a spread of \mathbb{F}_q^n for $n = rk$.

Definition 7 ([13, Definition 2]). We call *spread codes* of $\mathfrak{G}_{\mathbb{F}_q}(k, n)$ the subsets $S_r \subset \mathfrak{G}_{\mathbb{F}_q}(k, n)$ from Theorem 6.

Remark 8. Notice that

$$S_r = \{\text{rowsp}(A_1 \quad \dots \quad A_r) \in \mathfrak{G}_{\mathbb{F}_q}(k, n) \mid A_i \in \mathbb{F}_q[P] \text{ for all } i \in \{1, \dots, r\}\}.$$

In order to have a unique representative for the elements of S_r , we bring the matrices $(A_1 \quad \dots \quad A_r)$ in row reduced echelon form.

Lemma 9 ([13, Theorem 1]). *Let $S_r \subset \mathfrak{G}_{\mathbb{F}_q}(k, n)$ be a spread code. Then*

1. $d(\mathcal{U}, \mathcal{V}) = d_{\min}(S_r) = 2k$, for all $\mathcal{U}, \mathcal{V} \in S_n$ distinct, i.e., the code has maximal minimum distance, and
2. $|S_r| = \frac{q^n - 1}{q^k - 1}$, i.e., the code has maximal cardinality with respect to the given minimum distance.

Remark 10. In [17] the authors show that spread codes are an example of *orbit codes*. Moreover, in [18] it is shown that some spread codes are cyclic orbit codes under the action of the cyclic group generated by the companion matrix of a primitive polynomial.

Definition 11. A vector space $\mathcal{R} \in \mathfrak{G}_{\mathbb{F}_q}(\tilde{k}, rk)$ is *uniquely decodable* by the spread code $S_r \subset \mathfrak{G}_{\mathbb{F}_q}(k, n)$ if

$$\text{there exists a } \mathcal{C} \in S_r \text{ such that } d(\mathcal{R}, \mathcal{C}) < \frac{d_{\min}(S_r)}{2} = k.$$

In Section 3 we devise a minimum-distance decoding algorithm for uniquely decodable received spaces.

2.1. FURTHER NOTATIONS. We introduce in this subsection the notation we use in the paper.

Definition 12. Let $s \in \mathbb{N}$ with $s < k$ and denote by $L_{\mathbb{F}_{q^n}}^s \subset \mathbb{F}_{q^n}[x]$ the set of linearized polynomials of degree less than q^s . Equivalently, $f \in L_{\mathbb{F}_{q^n}}^s$ if and only if $f = \sum_{i=0}^{s-1} f_i x^{q^i}$ for some $f_i \in \mathbb{F}_{q^n}$.

In the rest of the work we denote q -th power exponents such as x^{q^i} with $x^{[i]}$.

Let \mathbb{F}_q be a finite field with q elements, and let $p \in \mathbb{F}_q[x]$ be a monic irreducible polynomial of degree $k > 1$. $P \in GL_k(\mathbb{F}_q)$ denotes the companion matrix of p , and $S \in GL_k(\mathbb{F}_{q^k})$ is a matrix which diagonalizes P .

We denote by $\Delta(x) := \text{diag}(x, x^{[1]}, \dots, x^{[k-1]}) \in \mathbb{F}_q[x]^{k \times k}$ a diagonal matrix, whose entry in position $(i+1, i+1)$ is $x^{[i]}$ for $i = 0, \dots, k-1$.

Let M be a matrix of size $k \times k$ and let $J = (j_1, \dots, j_s)$, $L = (l_1, \dots, l_s) \in \{1, \dots, k\}^s$. $[J; L]_M$ denotes the minor of the matrix M corresponding to the submatrix $(J; L)_M$ with row indices j_1, \dots, j_s and column indices l_1, \dots, l_s . We skip the suffix M when the matrix is clear from the context.

We introduce some operations on tuples. Let $K = (i_1, \dots, i_s) \in \{1, \dots, k\}^s$.

- $i \in K$ means that $i \in \{i_1, \dots, i_s\}$.
- $L \subset K$ means that $L = (i_{l_1}, \dots, i_{l_k})$ for $1 \leq l_1 < \dots < l_k \leq s$.
- $|K| := s$ is the length of the tuple.
- $K \cap J$ denotes the $L \subset K, J$ such that $|L|$ is maximal.
- If $J = (j_1, \dots, j_r)$ then $I \cup J := (i_1, \dots, i_s, j_1, \dots, j_r)$, i.e., \cup denotes the concatenation of tuples.
- If $J \subset K$ then $K \setminus J$ denotes the $L \subset K$ with $|L|$ maximal such that $J \cap L = \emptyset$ where \emptyset is the empty tuple.
- $\min K = \min\{i \mid i \in K\}$, with the convention that $\min \emptyset > \min K$ for any K .

We define the *non diagonal rank* of a matrix as follows.

Definition 13. Let $N \in \mathbb{F}_q^{k \times k}$. We define the *non diagonal rank* of N as

$$\text{ndrank}(N) := \min\{t \in \mathbb{N} \mid [J, L]_N = 0 \text{ for all } J, L \in \{1, \dots, k\}^t, J \cap L = \emptyset\} - 1.$$

At last, algorithms' complexities are expressed as $\mathcal{O}(\mathbb{F}; p(n, k))$, which corresponds to performing $\mathcal{O}(p(n, k))$ operations over a field \mathbb{F} , where n, k are given parameters.

2.2. RELATION WITH REED-SOLOMON-LIKE CODES. Reed-Solomon-like codes, also called lifted rank-metric codes, are a class of constant-dimension codes introduced in [8]. They are strictly related to maximal rank distance codes as introduced in [4]. We give here an equivalent definition of these codes.

Definition 14. Let $\mathbb{F}_q \subset \mathbb{F}_{q^n}$ be finite fields. Fix some \mathbb{F}_q -linearly independent elements $\alpha_1, \dots, \alpha_k \in \mathbb{F}_{q^n}$. Let $\psi : \mathbb{F}_{q^n} \rightarrow \mathbb{F}_q^n$ be an isomorphism of \mathbb{F}_q -vector spaces. A *Reed-Solomon-like (RSL) code* is defined as

$$RSL_{\mathbb{F}_{q^n}}^s := \left\{ \text{rowsp} \left(\begin{array}{c} \psi(f(\alpha_1)) \\ I \\ \vdots \\ \psi(f(\alpha_k)) \end{array} \right) \mid f \in L_{\mathbb{F}_{q^n}}^s \right\} \subseteq \mathfrak{G}_{\mathbb{F}_q}(k, k+n).$$

The following proposition establishes a relation between spread codes and RSL codes. The proof is easy, but rather technical, hence we omit it.

Proposition 15. Let $n = rk$, $\mathbb{F}_q \subseteq \mathbb{F}_{q^k} \subseteq \mathbb{F}_{q^n}$ finite fields, and $P \in GL_k(\mathbb{F}_q)$ the companion matrix of a monic irreducible polynomial $p \in \mathbb{F}_q[x]$ of degree $k > 0$. Let $\lambda \in \mathbb{F}_{q^k}$ be a root of p , $\mu_1, \dots, \mu_r \in \mathbb{F}_{q^n}$ a basis of \mathbb{F}_{q^n} over \mathbb{F}_{q^k} . Moreover, let $\psi : \mathbb{F}_{q^n} \rightarrow \mathbb{F}_q^n$ be the isomorphism of \mathbb{F}_q -vector spaces which maps the basis $(\lambda^i \mu_j)_{\substack{0 \leq j \leq k-1 \\ 1 \leq i \leq r}}$ to the standard basis of \mathbb{F}_{q^n} over \mathbb{F}_q . Then for every choice of $A_0, \dots, A_{r-1} \in \mathbb{F}_q[P]$ there exists a unique linearized polynomial of the form $f = ax$ with $a \in \mathbb{F}_{q^n}$ such that

$$(A_0 \ \cdots \ A_{r-1}) = \begin{pmatrix} \psi(f(1)) \\ \psi(f(\lambda)) \\ \vdots \\ \psi(f(\lambda^{k-1})) \end{pmatrix}.$$

The constant a is $a = \psi^{-1}(v)$ where $v \in \mathbb{F}_q^n$ is the first row of $(A_0 \ \cdots \ A_{r-1})$.

The proposition allows us to relate our spread codes to some RSL codes. The following corollary makes the connection explicit. We use the notation of Proposition 15.

Corollary 16. For each $1 \leq i \leq r-1$, let $\mu_{1,i}, \dots, \mu_{r-i,i}$ be a basis of $\mathbb{F}_{q^{(r-i)k}}$ over \mathbb{F}_{q^k} . Let $\psi_i : \mathbb{F}_{q^{(r-i)k}} \rightarrow \mathbb{F}_q^{(r-i)k}$ denote the isomorphism of vector spaces that maps the basis $(\lambda^j \mu_{l,i})_{\substack{0 \leq j \leq k-1 \\ 1 \leq l \leq r-i}}$ to the standard basis of $\mathbb{F}_q^{(r-i)k}$. Then

$$S_r = \bigcup_{i=1}^{r-1} \left\{ \text{rowsp} \left(\begin{array}{cc} & \psi_i(f(1)) \\ \underbrace{0 \ \cdots \ 0}_{i-1 \text{ times}} & I \\ & \vdots \\ & \psi_i(f(\lambda^{k-1})) \end{array} \right) \mid f = ax, \ a \in \mathbb{F}_{q^{(r-i)k}} \right\} \\ \cup \left\{ \left(\underbrace{0 \ \cdots \ 0}_{r-1 \text{ times}} \ I \right) \right\}.$$

Corollary 16 readily follows from Proposition 15.

The connection that we have established with RSL codes allows us to extend any minimum-distance decoding algorithm for RSL codes to a minimum-distance decoding algorithm for spread codes. We start with a key lemma.

Lemma 17. Let S_r be a spread code, and $\mathcal{R} = \text{rowsp}(R_1 \ \cdots \ R_r) \in \mathfrak{G}_{\mathbb{F}_q}(\tilde{k}, n)$ for some $\tilde{k} \leq k$. Assume there exists a $\mathcal{C} = \text{rowsp}(C_1 \ \cdots \ C_r) \in S_r$ such that $d(\mathcal{R}, \mathcal{C}) < k$. Let

$$i := \min \left\{ j \in \{1, \dots, r\} \mid \text{rank}(R_j) > \frac{\tilde{k} - 1}{2} \right\}.$$

It holds that:

- $C_j = 0$ for $1 \leq j < i$,
- $C_i = I$, and
- $d(\text{rowsp}(R_i \ R_{i+1} \ \cdots \ R_r), \text{rowsp}(I \ C_{i+1} \ \cdots \ C_r)) < k$.

Proof. The result follows from Lemma 28 and the observation that

$$d(\mathcal{C}, \mathcal{R}) \geq d(\text{rowsp}(C_i \ \cdots \ C_r), \text{rowsp}(R_i \ \cdots \ R_r)). \quad \square$$

In the next proposition, we use Corollary 16 and Lemma 17 to adapt to spread codes any decoding algorithm for RSL codes. In particular, we apply our results to the algorithms contained in [8] and [15], and we give the complexity of the resulting algorithms for spread codes.

Proposition 18. *Any minimum-distance decoding algorithm for RSL codes may be extended to a minimum-distance decoding algorithm for spread codes. In particular, the algorithms described in [8] and [15] can be extended to minimum-distance decoding algorithms for spread codes, with complexities $\mathcal{O}(\mathbb{F}_{q^{n-k}}; n^2)$ for the former and $\mathcal{O}(\mathbb{F}_{q^{n-k}}; k(n-k))$ for the latter.*

Proof. Suppose we are given a minimum-distance decoding algorithm for RSL codes. We construct a minimum-distance decoding algorithm for spread codes as follows: Let $\mathcal{R} = \text{rowsp}(R_1 \ \cdots \ R_r) \in \mathfrak{S}_{\mathbb{F}_q}(k, n)$ be the received word, and assume that there exists a $\mathcal{C} = \text{rowsp}(C_1 \ \cdots \ C_r) \in \mathbb{S}_r$ such that $d(\mathcal{R}, \mathcal{C}) < k$. First, one computes the rank of R_1, R_2, \dots until one finds an i such that $\text{rank}(R_i) > (k-1)/2$, $\text{rank}(R_j) \leq (k-1)/2$ for $j < i$. Thanks to Lemma 17, one knows that $C_j = 0$ for $j < i$ and $C_i = I$. Moreover, one has

$$d(\text{rowsp}(R_i \ R_{i+1} \ \cdots \ R_r), \text{rowsp}(I \ C_{i+1} \ \cdots \ C_r)) < k.$$

Therefore, one can apply the minimum-distance decoding algorithm for RSL codes to the received word $\text{rowsp}(R_i \ R_{i+1} \ \cdots \ R_r)$ in order to compute C_{i+1}, \dots, C_r .

Assume now that one uses as minimum-distance decoder for RSL codes either the decoding algorithm from [8], or the one from [15]. The complexity of computing the rank of R_1, \dots, R_i by computing row reduced echelon forms is $\mathcal{O}(\mathbb{F}_q; nk^2)$. The complexity of the decoding algorithm for RSL codes is $\mathcal{O}(\mathbb{F}_{q^{n-k}}; n^2)$ for the one in [8] and $\mathcal{O}(\mathbb{F}_{q^{n-k}}; k(n-k))$ for the one in [15]. The complexity of the decoding algorithm is the dominant term in the complexity estimate. \square

It is well known that RSL codes are strictly related to the rank-metric codes introduced in [4]. Although the rank metric on rank-metric codes is equivalent to the subspace distance on RSL codes, the minimum-distance decoding problem in the former is not equivalent to the one in the latter. In [15] the authors introduced the *Generalized Decoding Problem for Rank-Metric Codes*, which is equivalent to the minimum-distance decoding problem of RSL codes. Decoding algorithms for rank-metric codes such as the ones contained in [4, 11, 14] must be generalized in order to be effective for the *Generalized Decoding Problem for Rank-Metric Codes*, and consequently, to be applicable to RSL codes.

Another interesting application of Lemma 17 allows us to improve the efficiency of the decoding algorithm for the codes proposed in [16]. For the relevant definitions, we refer the interested reader to the original article.

Corollary 19. *There is an algorithm which decodes the codes from [16] and has complexity $\mathcal{O}(\mathbb{F}_{q^{n-k}}; k(n-k))$.*

The algorithm is a combination of Lemma 17 and the decoding algorithm contained in [15]. First, by Lemma 17, one finds the position of the identity matrix. This reduces the minimum-distance decoding problem to decoding a RSL code, so one can use the algorithm from [15].

3. THE MINIMUM-DISTANCE DECODING ALGORITHM

In this section we devise a new minimum-distance decoding algorithm for spread codes. In the next section, we show that our algorithm is more efficient than the ones present in the literature, when $n \gg k$.

We start by proving some results on matrices, which we will be used to design and prove the correctness of the decoding algorithm.

3.1. PRELIMINARY RESULTS ON MATRICES. Let \mathbb{F} be a field and let $m \in \mathbb{F}[y_1, \dots, y_s]$ be a polynomial of the form $m = \sum_{U \subseteq (1, \dots, s)} a_U y_U$ where $y_U := \prod_{u \in U} y_u$, $a_{(1, \dots, s)} \neq 0$.

Lemma 20. *The following are equivalent:*

1. *The polynomial m decomposes in linear factors, i.e.,*

$$m = a_{(1, \dots, s)} \prod_{u \in (1, \dots, s)} (y_u + \mu_u)$$

$$\text{where } \mu_u = \frac{a_{(1, \dots, s) \setminus \{u\}}}{a_{(1, \dots, s)}} \in \mathbb{F}.$$

2. *It holds*

$$(2) \quad a_U a_V = a_{U \cap V} a_{(1, \dots, s)}$$

for all U, V such that $|V| = s - 1$ and

$$\min((1, \dots, s) \setminus V) < \min((1, \dots, s) \setminus U).$$

Proof. We proceed by induction on s .

\Rightarrow : If $s = 1$, m is a linear polynomial. Let us now suppose the thesis is true for $s - 1$. Then

$$a_{(1, \dots, s)} \prod_{u \in (1, \dots, s)} (y_u + \mu_u) = a_{(1, \dots, s)} (y_s + \mu_s) \left(\sum_{U \subseteq (1, \dots, s-1)} \tilde{a}_U y_U \right)$$

where $\tilde{a}_{(1, \dots, s-1)} = 1$ and the coefficients \tilde{a}_U with $U \subseteq (1, \dots, s-1)$ satisfy by hypothesis condition (2). The coefficients of m are $a_U = \tilde{a}_{U \setminus \{s\}}$ if $s \in U$, and $a_U = \mu_s \tilde{a}_U$ otherwise. Therefore we only need to prove that (2) holds for $U \in (1, \dots, s-1)$. The equality is $a_{(1, \dots, s)} a_U = a_U a_{(1, \dots, s)}$ hence it is trivial.

\Leftarrow : The thesis is trivial for $s = 1$. Let us assume that the thesis holds for $s - 1$.

We explicitly show the extraction of a linear factor of the polynomial.

$$\begin{aligned} m &= \sum_{U \subseteq (1, \dots, s)} a_U y_U = \sum_{\substack{U \subseteq (1, \dots, s) \\ 1 \in U}} (a_U y_U + a_{U \setminus \{1\}} y_{U \setminus \{1\}}) \\ &= \sum_{\substack{U \subseteq (1, \dots, s) \\ 1 \in U}} \left(a_U y_1 y_{U \setminus \{1\}} + \frac{a_U a_{(2, \dots, s)}}{a_{(1, \dots, s)}} y_{U \setminus \{1\}} \right) \end{aligned}$$

$$= \left(y_1 + \frac{a_{(2,\dots,s)}}{a_{(1,\dots,s)}} \right) \cdot \left(\sum_{\substack{U \subseteq \{1,\dots,s\} \\ 1 \in U}} a_U y_{U \setminus \{1\}} \right).$$

The thesis is true by induction. \square

Let $\mathbb{F}[x_{i,j}]_{1 \leq i,j \leq k}$ be a polynomial ring with coefficients in a field \mathbb{F} . Consider the generic matrix of size $k \times k$

$$M := \begin{pmatrix} x_{1,1} & \cdots & x_{1,k} \\ \vdots & & \vdots \\ x_{k,1} & \cdots & x_{k,k} \end{pmatrix}.$$

Denote by $\mathcal{I}_{s+1} \subset \mathbb{F}[x_{i,j}]_{1 \leq i,j \leq k}$ the ideal generated by all minors of size $s+1$ of M , which do not involve entries on the diagonal, i.e.,

$$\mathcal{I}_{s+1} := ([J, L] \mid J, L \in \{1, \dots, k\}^{s+1}, J \cap L = \emptyset).$$

We establish some relations on the minors of M , modulo the ideal \mathcal{I}_{s+1} .

Lemma 21. *Let $J = (j_1, \dots, j_k), L = (l_1, \dots, l_k) \in \{1, \dots, k\}^k$, $J_s = (j_1, \dots, j_s)$, and $L_s = (l_1, \dots, l_s)$. Then*

$$[J_s; L_s][J; L] = \sum_{t=s+1}^k (-1)^{t+s+1} [J_s \cup (j_t); L_s \cup (l_{s+1})][J \setminus (j_t); L \setminus (l_{s+1})].$$

Proof. Notice that if we consider as convention that $[\emptyset; \emptyset] = 1$, i.e., when $s = 0$, we get the determinant formula.

We proceed by induction on s . Let us consider the case when $s = 1$, i.e., $[J_1; L_1] = (x_{j_1, l_1})$. Then,

$$\begin{aligned} (x_{j_1, l_1}) [J; L] &= \sum_{t=1}^k (-1)^{t+2} x_{j_1, l_1} x_{j_t, l_2} [J \setminus (j_t); L \setminus (l_2)] \\ &= -x_{j_1, l_1} x_{j_1, l_2} [J \setminus (j_1); L \setminus (l_2)] \\ &\quad + \sum_{t=2}^k (-1)^{t+2} [(j_1, j_t); (l_1, l_2)] + x_{j_t, l_1} x_{j_1, l_2} [J \setminus (j_t); L \setminus (l_2)] \\ &= \sum_{t=2}^k (-1)^{t+2} [(j_1, j_t); (l_1, l_2)] [J \setminus (j_t); L \setminus (l_2)] \\ &\quad + x_{j_1, l_2} [J; (l_1, l_1, l_3, \dots, l_k)]. \end{aligned}$$

For $s = 1$ the thesis is true because $[J; (l_1, l_1, l_3, \dots, l_k)] = 0$ since column l_1 appears twice.

Assume that the thesis is true for $s - 1$.

$$[J_s; L_s][J; L] = \sum_{t=1}^k (-1)^{t+s+1} x_{j_t, l_{s+1}} [J_s; L_s][J \setminus (j_t); L \setminus (l_{s+1})].$$

Let us now focus on the factor $x_{j_r, l_{s+1}} [J_s; L_s]$ for $r \geq s + 1$, we get

$$x_{j_r, l_{s+1}} [J_s; L_s] = [J_s \cup (j_r); L_s \cup (l_{s+1})] + \sum_{t=1}^s (-1)^{t+s} x_{j_t, l_{s+1}} [J_s \setminus (j_t) \cup (j_r); L_s].$$

By substitution it follows that

$$\begin{aligned}
 [J_s; L_s][J; L] &= \sum_{t=s+1}^k (-1)^{t+s+1} [J_s \cup (j_t); L_s \cup (l_{s+1})][J \setminus (j_t); L \setminus (l_{s+1})] + \\
 &\quad + \sum_{t=1}^s (-1)^{t+s+1} x_{j_t, l_{s+1}} \left([J_s; L_s][J \setminus (j_t); L \setminus (l_{s+1})] + \right. \\
 &\quad \left. + \sum_{r=s+1}^k (-1)^{r+s} [J_s \setminus (j_t) \cup (j_r); L_s][L \setminus (j_r); L \setminus (l_{s+1})] \right) \\
 &= \sum_{t=s+1}^k (-1)^{t+s+1} [J_s \cup (j_t); L_s \cup (l_{s+1})][J \setminus (j_t); L \setminus (l_{s+1})] + \\
 &\quad + \sum_{t=1}^s (-1)^{t+s+1} x_{j_t, l_{s+1}} ([J_s \setminus (j_t); L_s \setminus (l_s)][J; \bar{L}])
 \end{aligned}$$

where $\bar{L} = (l_1, \dots, l_s, l_s, l_{s+2}, \dots, l_k)$. The repetition of column l_s twice in \bar{L} implies that $[J; \bar{L}] = 0$. The last equality follows from the induction hypothesis. \square

The following is an easy consequence of Lemma 21.

Proposition 22. *Let $J, L \subset K = (1, \dots, k)$ such that $J \cap L = \emptyset$. Then*

$$[J, L][K, K] - [J \cup (i); L \cup (i)][K \setminus (i); K \setminus (i)] = \sum_{l \in K \setminus (J \cup (i))} h_l [J \cup (i), L \cup (l)] \in \mathcal{I}_{s+1},$$

with $h_l \in \mathbb{F}[x_{i,j}]_{1 \leq i, j \leq k}$ for any $l \in K \setminus (J \cup (i))$.

We now study the minors of a matrix of the form $S^{-1}NS$ where $N \in \mathbb{F}_q^{k \times k}$ and S has a special form, which we describe in the next lemma.

Lemma 23. *Let $P \in GL_k(\mathbb{F}_q)$ to be the companion matrix of a monic irreducible polynomial $p \in \mathbb{F}_q$ of degree $k > 0$, and let $\lambda \in \mathbb{F}_{q^k}$ be a root of p . Then the matrix*

$$(3) \quad S := \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \lambda & \lambda^{[1]} & \lambda^{[2]} & \cdots & \lambda^{[k-1]} \\ \lambda^2 & \lambda^{2 \cdot [1]} & \lambda^{2 \cdot [2]} & \cdots & \lambda^{2 \cdot [k-1]} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda^{k-1} & \lambda^{(k-1) \cdot [1]} & \lambda^{(k-1) \cdot [2]} & \cdots & \lambda^{(k-1) \cdot [k-1]} \end{pmatrix}.$$

diagonalizes P .

Proof. The eigenvalues of the matrix P correspond to the roots of the irreducible polynomial $p \in \mathbb{F}_q[x]$. If $\lambda \in \mathbb{F}_{q^k}$ is an element such that $p(\lambda) = 0$, then $p = \prod_{i=0}^{k-1} (x - \lambda^{[i]})$ by [10, Theorem 2.4]. It is enough to show that the columns of S correspond to the eigenvectors of P . Let $i \in \{0, \dots, k-1\}$, then

$$P \begin{pmatrix} 1 \\ \lambda^{[i]} \\ \vdots \\ \lambda^{(k-1) \cdot [i]} \end{pmatrix} = \begin{pmatrix} \lambda^{[i]} \\ \lambda^{2 \cdot [i]} \\ \vdots \\ -\sum_{j=0}^{k-1} p_j \lambda^{j \cdot [i]} \end{pmatrix} = \begin{pmatrix} \lambda^{[i]} \\ \lambda^{2 \cdot [i]} \\ \vdots \\ \left(-\sum_{j=0}^{k-1} p_j \lambda^j \right)^{[i]} \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^{[i]} \\ \lambda^{2 \cdot [i]} \\ \vdots \\ \lambda^{k \cdot [i]} \end{pmatrix} = \lambda^{[i]} \begin{pmatrix} 1 \\ \lambda^{[i]} \\ \vdots \\ \lambda^{(k-1) \cdot [i]} \end{pmatrix}. \quad \square$$

We now establish some properties of S .

Lemma 24. *The matrices S and S^{-1} defined by (3) satisfy the following properties:*

1. *the entries of the first column of S (respectively, the first row of S^{-1}) form a basis of \mathbb{F}_{q^k} over \mathbb{F}_q , and*
2. *the entries of the $(i+1)$ -th column of S (respectively, row of S^{-1}) are the q -th power of the ones of the i -th column (respectively, row) for $i = 1, \dots, k-1$.*

Proof. The two properties for the matrix S come directly from its definition. By [10, Definition 2.30] we know that there exists a unique basis $\{\gamma_0, \dots, \gamma_{k-1}\}$ of \mathbb{F}_{q^k} over \mathbb{F}_q such that

$$\mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(\lambda^i \gamma_j) = \begin{cases} 1, & i = j; \\ 0, & i \neq j, \end{cases}$$

where $\mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(\alpha) := 1 + \alpha^{[1]} + \dots + \alpha^{[k-1]}$ for $\alpha \in \mathbb{F}_{q^k}$. We have

$$S^{-1} = \begin{pmatrix} \gamma_0 & \gamma_1 & \dots & \gamma_{k-1} \\ \gamma_0^{[1]} & \gamma_1^{[1]} & \dots & \gamma_{k-1}^{[1]} \\ \vdots & \vdots & & \vdots \\ \gamma_0^{[k-1]} & \gamma_1^{[k-1]} & \dots & \gamma_{k-1}^{[k-1]} \end{pmatrix}. \quad \square$$

The next theorem and corollary will be used in Subsection 3.3 to devise a simplified minimum-distance decoding algorithm, under the assumption that the first k columns of the received vector space are linearly independent.

Theorem 25. *Let $t \leq k$ and let $N \in \mathbb{F}_q^{t \times k}$ and $S \in \mathbb{F}_{q^k}^{k \times t}$ be two matrices satisfying the following properties:*

- *N has full rank,*
- *the entries of the first column of S form a basis of \mathbb{F}_{q^k} over \mathbb{F}_q , and*
- *the entries of the $(i+1)$ -th column of S are the q -th power of the ones of the i -th column, for $i = 1, \dots, t-1$.*

Then $NS \in GL_t(\mathbb{F}_{q^k})$.

Proof. Let

$$N := (n_{ij})_{\substack{1 \leq i \leq t \\ 1 \leq j \leq k}} \text{ and } NS = (t_{ij})_{\substack{1 \leq i \leq t \\ 1 \leq j \leq t}}.$$

Let $S := (s_{ij})_{\substack{1 \leq i \leq k \\ 1 \leq j \leq t}} = \left(s_i^{[j-1]} \right)_{\substack{1 \leq i \leq k \\ 1 \leq j \leq t}}$ where $s_1, \dots, s_k \in \mathbb{F}_{q^k}$ form a basis of \mathbb{F}_{q^k} over \mathbb{F}_q . Then:

$$t_{ij} := \sum_{l=1}^k n_{il} s_{lj} = \sum_{l=1}^k n_{il} s_l^{[j-1]} = \left(\sum_{l=1}^k n_{il} s_l \right)^{[j-1]},$$

since the entries of N are in \mathbb{F}_q . Let $\tau_i := \sum_{l=1}^k n_{il}s_l \in \mathbb{F}_{q^k}$, then

$$NS = \begin{pmatrix} \tau_1 & \tau_1^{[1]} & \dots & \tau_1^{[t-1]} \\ \tau_2 & \tau_1^{[1]} & \dots & \tau_2^{[t-1]} \\ \vdots & \vdots & & \vdots \\ \tau_r & \tau_r^{[1]} & \dots & \tau_r^{[t-1]} \end{pmatrix}.$$

The elements $\tau_1, \dots, \tau_t \in \mathbb{F}_{q^k}$ are linearly independent over \mathbb{F}_q . Indeed, the linear combination

$$\sum_{i=1}^k \alpha_i \tau_i = \sum_{i=1}^t \alpha_i \sum_{l=1}^k n_{il}s_l = \sum_{l=1}^k \left(\sum_{i=1}^t \alpha_i n_{il} \right) s_l$$

is zero only when $\sum_{i=1}^t \alpha_i n_{il} = 0$ for $l = 1, \dots, t$. Since N has full rank it follows that $\alpha_1, \dots, \alpha_t$ must all be zero, leading to the linear independence of τ_1, \dots, τ_t .

Now let $a_0, \dots, a_{t-1} \in \mathbb{F}_{q^k}$ be such that

$$NS \begin{pmatrix} a_0 \\ \vdots \\ a_{t-1} \end{pmatrix} = 0,$$

and consider the linearized polynomial $f = \sum_{i=0}^{t-1} a_i x^{[t-i]}$. The elements τ_1, \dots, τ_t are by assumption roots of f . Since f is a linear map, the kernel of f contains the subspace $\langle \tau_1, \dots, \tau_t \rangle \subset \mathbb{F}_{q^k}$. Therefore f is a polynomial of degree q^{t-1} with q^t different roots, then $a_0 = \dots = a_{t-1} = 0$. \square

Corollary 26. *Let $S \in GL_k(\mathbb{F}_{q^k})$ be the matrix specified in (3) and $N \in \mathbb{F}_q^{k \times k}$. Then for any $J, L \subset (1, \dots, k)$ tuples of consecutive indices with $|J| = |L| = \text{rank}(N)$, one has $[J; L]_{S^{-1}NS} \neq 0$.*

Proof. Let $t = \text{rank}(N)$ and $J, L \subset (1, \dots, k)$ with $|J| = |L| = t$, let $H = (1, \dots, t)$. Let $N_1 \in \mathbb{F}_q^{k \times t}$ and $N_2 \in \mathbb{F}_q^{t \times k}$ be matrices with full rank such that $N = N_1 N_2$. One has

$$[J, L]_{S^{-1}NS} = [J, L]_{S^{-1}N_1 \cdot N_2 S} = [J, H]_{S^{-1}N_1} [H, L]_{N_2 S}.$$

We can now focus on the characterization of the maximal minors of the matrix $N_2 S$. The following considerations will also work for the matrix $S^{-1}N_1$ considering its transpose.

The minor $[H, L]_{N_2 S}$ is the determinant of a square matrix obtained by multiplying N_2 with the submatrix consisting of the columns of S indexed by L . Let L contain consecutive indices. By Lemma 24, the submatrix of S that we obtain together with N_2 satisfy the conditions of Theorem 25. It follows that $[H, L]_{N_2 S} \neq 0$.

As a consequence we have that $[J, L]_{S^{-1}NS} \neq 0$ when both J and L are tuples of consecutive indices. \square

The following is a reformulation of Corollary 26 for small rank matrices.

Corollary 27. *Let $N \in \mathbb{F}_q^{k \times k}$ be a matrix such that $\text{rank}(N) \leq \frac{k-1}{2}$ and $S \in GL_k(\mathbb{F}_{q^k})$ defined as in (3). Then for any choice $J, L \subset (1, \dots, k)$ of consecutive indices with $|J| = |L| = \text{rank}(N)$,*

$$[J, L]_{S^{-1}NS} \neq 0.$$

In particular,

$$\text{ndrank}(S^{-1}NS) = \text{rank}(N).$$

3.2. THE DECODING ALGORITHM. In this subsection we devise an efficient minimum-distance decoding algorithm for spread codes, and establish some closely related mathematical results.

We start by reducing the minimum-distance decoding algorithm for S_r to at most $r - 1$ instances of the minimum-distance decoding algorithm for S_2 . Notice that the minimum-distance decoders for the case $r = 2$ can be run in parallel.

Let $\mathcal{R} = \text{rowsp}(R_1 \ \cdots \ R_r)$ be a received space. We assume that

$$1 \leq \tilde{k} = \text{rank}(\mathcal{R}) \leq k.$$

Algorithm 1 on page 456 is based on the following lemma.

Lemma 28. *Let S_r be a spread code, and $\mathcal{R} = \text{rowsp}(R_1 \ \cdots \ R_r) \in \mathfrak{S}_{\mathbb{F}_q}(\tilde{k}, rk)$. Assume there exists a $\mathcal{C} = \text{rowsp}(C_1 \ \cdots \ C_r) \in \mathcal{S}$ such that $d(\mathcal{R}, \mathcal{C}) < k$. It holds*

$$C_i = 0 \iff \text{rank}(R_i) \leq \frac{\tilde{k} - 1}{2}.$$

Proof. \Rightarrow : Let $i \in \{1, \dots, r\}$ be an index such that $C_i = 0$. By the construction of a spread code there exists a $j \in \{1, \dots, r\}$ with $C_j = I$. We claim that $\dim(\mathcal{C} \cap \mathcal{R}) > \frac{\tilde{k}}{2}$. In fact,

$$k > \dim \mathcal{C} + \dim \mathcal{R} - 2 \dim(\mathcal{C} \cap \mathcal{R}) = k + \tilde{k} - 2 \dim(\mathcal{C} \cap \mathcal{R}).$$

From the claim it follows that

$$\text{rank} \begin{pmatrix} 0 & I \\ R_i & R_j \end{pmatrix} \leq \text{rank} \begin{pmatrix} C_1 & \cdots & C_r \\ R_1 & \cdots & R_r \end{pmatrix} = k + \tilde{k} - \dim(\mathcal{C} \cap \mathcal{R}) < k + \frac{\tilde{k}}{2}.$$

This proves that

$$\text{rank}(R_i) < \frac{\tilde{k}}{2}.$$

\Leftarrow : Let $i \in \{1, \dots, r\}$ be such that $\text{rank}(R_i) \leq \frac{\tilde{k}-1}{2}$ and assume by contradiction that $C_i \in \mathbb{F}_q[P]^*$. It follows that

$$\dim(\mathcal{C} \cap \mathcal{R}) \leq \dim(\text{rowsp}(C_i) \cap \text{rowsp}(R_i)) = \dim(\text{rowsp}(R_i)) \leq \frac{\tilde{k} - 1}{2}$$

which contradicts the assumption that $d(\mathcal{C}, \mathcal{R}) = k + \tilde{k} - 2 \dim(\mathcal{C} \cap \mathcal{R}) < k$. \square

Because of Lemma 28, we may now focus on designing a minimum-distance decoding algorithm for the case where $n = 2k$. For the remainder of this subsection, we consider the spread code

$$\mathcal{S} = \mathcal{S}_2 = \{\text{rowsp}(I \ A) \mid A \in \mathbb{F}_q[P]\} \cup \{\text{rowsp}(0 \ I)\}$$

where I and 0 are respectively the identity and the zero matrix of size $k \times k$.

Since a minimum-distance decoding algorithm decodes uniquely up to half the minimum distance, we are interested in writing an algorithm with the following specifications:

Input: $\mathcal{R} = \text{rowsp}(R_1 \ R_2) \in \mathfrak{S}_{\mathbb{F}_q}(\tilde{k}, 2k)$,

$P \in GL_k(\mathbb{F}_q)$ the companion matrix of $p \in \mathbb{F}_q[x]$ and

$S \in GL_k(\mathbb{F}_{q^k})$ its diagonalizing matrix.

Output: $\mathcal{C} \in \mathcal{S} \subset \mathfrak{S}_{\mathbb{F}_q}(k, 2k)$ such that $d(\mathcal{R}, \mathcal{C}) < \frac{d(\mathcal{S})}{2} = k$, if such a \mathcal{C} exists.

Algorithm 1: Minimum-distance decoding algorithm: $n = rk$, $r > 2$

input : $\mathcal{R} = \text{rowsp}(R_1 \ \cdots \ R_r) \in \mathfrak{G}_{\mathbb{F}_q}(\tilde{k}, rk)$, $r > 2$,
 $P \in GL_k(\mathbb{F}_q)$ the companion matrix of $p \in \mathbb{F}_q[x]$ and
 $S \in GL_k(\mathbb{F}_{q^k})$ its diagonalizing matrix.
output: $\mathcal{C} \in \mathcal{S}_r \subset \mathfrak{G}_{\mathbb{F}_q}(k, rk)$ such that $d(\mathcal{R}, \mathcal{C}) < k$, if such a \mathcal{C} exists.
Let $r_i = \text{rank}(R_i)$ for $i = 1, \dots, r$;
if $r_i \leq \frac{\tilde{k}-1}{2}$ for all $i \in \{1, \dots, r\}$ **then**
| **return** there exists no $\mathcal{C} \in \mathcal{S}_r$ such that $d(\mathcal{R}, \mathcal{C}) < k$
end
Let $j = \min \{i \in \{1, \dots, r\} \mid r_i > \frac{\tilde{k}-1}{2}\}$;
for $i \in \{1, \dots, r\}$ and $r_i \leq \frac{\tilde{k}-1}{2}$ **do**
| $C_i = 0 \in \mathbb{F}_q^{k \times k}$;
end
for $j < i \leq r$ and $r_i > \frac{\tilde{k}-1}{2}$ **do**
| Run a minimum-distance decoding algorithm for $r = 2$ with input
| $\mathcal{R} = \text{rowsp}(R_j \ R_i)$, P and S ;
| **if** minimum-distance decoding algorithm returns no $\mathcal{C} \in \mathcal{S}_2$ **then**
| | **return** there exists no $\mathcal{C} \in \mathcal{S}_r$ such that $d(\mathcal{R}, \mathcal{C}) < k$;
| | **else** let $C_i \in \mathbb{F}_q[P]$ such that $\mathcal{C} = \text{rowsp}(I \ C_i)$;
| **end**
end
return $\mathcal{C} = \text{rowsp}(C_1 \ \cdots \ C_r)$.

We first give a membership criterion for spread codes. We follow the notation given at the beginning of this section.

Proposition 29 ([13, Lemma 5 and Corollary 6]). *Let $A \in GL_k(\mathbb{F}_q) \cup \{0\}$. The following are equivalent:*

1. $A \in \mathbb{F}_q[P]$.
2. $S^{-1}AS$ is a diagonal matrix.
3. $AP = PA$.

If this is the case, then $S^{-1}AS = \Delta(\lambda)$ for some $\lambda \in \mathbb{F}_{q^k}$.

From the proposition, we get an efficient algorithm to test whether a received vector space is error-free.

Corollary 30 ([13, Corollary 6], Membership Test). *Let $\mathcal{R} = \text{rowsp}(R_1 \ R_2) \in \mathfrak{G}_{\mathbb{F}_q}(k, 2k)$. Then $\mathcal{R} \in \mathcal{S}$ if and only if either $R_1 \in GL_k(\mathbb{F}_q)$ and $S^{-1}R_1^{-1}R_2S$ is diagonal or $R_1 = 0$ and $R_2 \in GL_k(\mathbb{F}_q)$.*

The following is an easy consequence of Lemma 28. It allows us to efficiently test whether the sent codeword was $\text{rowsp}(0 \ I)$, or $\text{rowsp}(I \ 0)$.

Corollary 31. *Let $\mathcal{R} = \text{rowsp}(R_1 \ R_2) \in \mathfrak{G}_{\mathbb{F}_q}(\tilde{k}, 2k)$ be a received space, and assume that it is uniquely decodable. The following are equivalent:*

- $\text{rank}(R_1) \leq \frac{\tilde{k}-1}{2}$, and
- the output of a minimum-distance decoding algorithm is $\text{rowsp}(0 \ I)$.

The analogous statement holds for R_2 .

Because of Corollary 31, we can restrict our decoding algorithm to look for codewords of the form $\mathcal{C} = \text{rowsp} \begin{pmatrix} I & A \end{pmatrix}$ where $A \in \mathbb{F}_q[P]$. Since there is an obvious symmetry in the construction of a spread code, we assume without loss of generality that

$$\text{rank}(R_1) \geq \text{rank}(R_2) > \frac{\tilde{k} - 1}{2}.$$

With the following theorem we translate the unique decodability condition into a rank condition, and then into a greatest common divisor condition.

Theorem 32. *Let $\mathcal{R} \in \mathfrak{G}_{\mathbb{F}_q}(\tilde{k}, n)$ be a subspace with*

$$\text{rank}(R_1) \geq \text{rank}(R_2) > \frac{\tilde{k} - 1}{2}.$$

The following are equivalent:

- \mathcal{R} is uniquely decodable.
- There exists a unique $\mu \in \mathbb{F}_{q^k}$ such that

$$(4) \quad \text{rank}(S^{-1}R_1S\Delta(\mu) - S^{-1}R_2S) \leq \frac{\tilde{k} - 1}{2}.$$

- $x - \mu = \gcd \left(\left\{ [J; L]_{S^{-1}R_1S\Delta(x) - S^{-1}R_2S} \mid J, L \in \{1, \dots, k\}^{\lfloor \frac{\tilde{k}+1}{2} \rfloor} \right\}, x^{[k]} - x \right),$
for some $\mu \in \mathbb{F}_{q^k}$.

Proof. \mathcal{R} is uniquely decodable if and only if there exists a unique matrix $X \in \mathbb{F}_q[P]$ such that

$$\begin{aligned} k - 1 \geq d(\mathcal{R}, \mathcal{C}) &= 2\text{rank} \begin{pmatrix} I & X \\ R_1 & R_2 \end{pmatrix} - (k + \tilde{k}) \\ &= 2\text{rank} \begin{pmatrix} I & X \\ 0 & R_1X - R_2 \end{pmatrix} - (k + \tilde{k}) = 2\text{rank}(R_1X - R_2) + k - \tilde{k}. \end{aligned}$$

Furthermore we get that $\text{rank}(R_1X - R_2) = \text{rank}(S^{-1}R_1S\Delta(x) - S^{-1}R_2S)$, where $S^{-1}XS = \Delta(x)$ is a consequence of Lemma 29. The existence of a unique solution $X \in \mathbb{F}_q[P]$ is then equivalent to the existence of a unique $\mu \in \mathbb{F}_{q^k}$ such that

$$\text{rank}(S^{-1}R_1S\Delta(\mu) - S^{-1}R_2S) \leq \frac{\tilde{k} - 1}{2}.$$

This is equivalent to the condition that all minors of size $\lfloor \frac{\tilde{k}+1}{2} \rfloor$ of $S^{-1}R_1S\Delta(\mu) - S^{-1}R_2S$ are zero. This leads to a nonempty system of polynomials in the variable x having a unique solution $\mu \in \mathbb{F}_{q^k}$. Therefore

$$x - \mu \mid \gcd \left(\left\{ [J; L]_{S^{-1}R_1S\Delta(x) - S^{-1}R_2S} \mid J, L \in \{1, \dots, k\}^{\lfloor \frac{\tilde{k}+1}{2} \rfloor} \right\}, x^{[k]} - x \right).$$

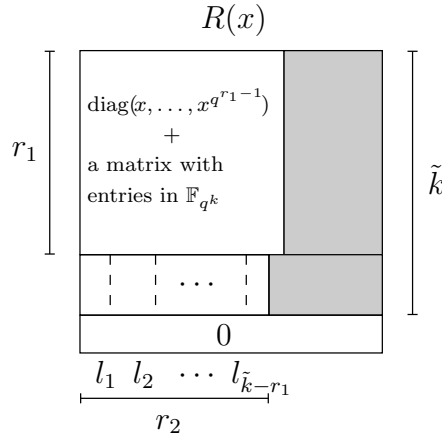
Equality follows from the uniqueness of μ . \square

Theorem 32 has the following immediate consequence, which constitutes a step forward towards the design of our decoding algorithm.

Corollary 33. *Assume that the received space $\mathcal{R} \in \mathfrak{G}_{\mathbb{F}_q}(\tilde{k}, n)$ is uniquely decodable. Then it decodes to*

$$\mathcal{C} = \text{rowsp} \begin{pmatrix} I & S\Delta(\mu)S^{-1} \end{pmatrix} \in \mathcal{S}$$

where $\mu \in \mathbb{F}_{q^k}$ is a root of all the minors of size $\lfloor \frac{\tilde{k}+1}{2} \rfloor$ of $S^{-1}R_1S\Delta(x) - S^{-1}R_2S$.


 FIGURE 2. Representation of the matrix $R(x)$.

$(l_1, \dots, l_{\tilde{k}-r_1}) \subset I$ is the tuple of indices of the columns corresponding to the pivots of $MS^{-1}R_2S$. Hence, for all $i \in \{1, \dots, \tilde{k} - r_1\}$ the entries of columns l_i of $R(x)$ are all zero except for the entry l_i , which is $x^{[l_i-1]}$, and the entry $r_1 + i$, which is 1.

Now consider the square submatrix $R'(x)$ of $R(x)$ of size $2r_1 - \tilde{k}$ defined by the rows and columns indexed by

$$I' := I \setminus (l_1, \dots, l_{\tilde{k}-r_1}, r_1 + 1, \dots, k).$$

The matrix $R'(x)$ is a matrix containing unknowns only in the diagonal entries.

Let $(J; L)_{R'(x)}$ be a submatrix of $R'(x)$, then it holds that

$$[J, L]_{R'(x)} = [J \cup (r_1 + 1, \dots, \tilde{k}), L \cup (l_1, \dots, l_{\tilde{k}-r_1})]_{R(x)}.$$

Let $\mu \in \mathbb{F}_{q^k}$ be the unique element satisfying condition (4), by the previous relation it holds that

$$(5) \quad \text{rank}(R'(\mu)) \leq \frac{\tilde{k} - 1}{2} - (\tilde{k} - r_1) = \frac{2r_1 - \tilde{k} - 1}{2}.$$

This implies that μ is a root of all $[J, L]_{R'(x)}$ such that $|J| = |L| = \lfloor \frac{\tilde{k}+1}{2} \rfloor - (\tilde{k} - r_1)$.

Let $J', L' \subset I'$ be tuples of indices such that

$$(6) \quad \begin{aligned} &J' \cap L' = \emptyset, [J', L']_{R'(x)} \neq 0, \text{ and} \\ &[J' \cup (j), L' \cup (l)]_{R'(x)} = 0 \text{ for any } j \neq l \in I' \setminus (J' \cup L'). \end{aligned}$$

The existence of a couple of tuples satisfying these conditions is ensured by the definition of $\text{ndrank}(R'(x))$.

Let $K \subset I' \setminus (J' \cup L')$ with $|K| = \lfloor \frac{\tilde{k}+1}{2} \rfloor - (\tilde{k} - r_1) - |J'|$. K is non empty since by (5)

$$|K| \geq \lfloor \frac{\tilde{k}+1}{2} \rfloor - (\tilde{k} - r_1) - \frac{2r_1 - \tilde{k} - 1}{2} = \lfloor \frac{\tilde{k}+1}{2} \rfloor - \frac{\tilde{k} - 1}{2} > 0.$$

Define $J := J' \cup K$ and $L := L' \cup K$.

Combining conditions (6) and Proposition 22 we obtain that

$$[J, L][I', I'] - [J \cup (i), L \cup (i)][I' \setminus (i), I' \setminus (i)] = 0$$

for $i \in K$. It follows by Lemma 20 that the polynomial $[J, L]$ factors as follows

$$[J, L]_{R(x)} = [J \setminus K, L \setminus K]_{R(0)} \prod_{i \in K} (x^{[i]} - \mu_i).$$

with $\mu_i = \frac{[J \setminus (i), L \setminus (i)]_{R(0)}}{[J \setminus K, L \setminus K]_{R(0)}}$ and $\mu \in \left\{ \mu_i^{[k-i]} \mid i \in K \right\}$. \square

Summarizing, the decoding algorithm that we obtain exploiting Theorem 34 is as follows:

1. Find tuples J, L satisfying the assumptions (6) of the theorem. Algorithm 4 in Section 4 gives an efficient way to find such tuples.
2. Write down the roots of the minor $[J, L]_{R(x)}$, where $R(x)$ is the matrix in the statement of the theorem. Theorem 34 gives explicit formulas for the roots, so this step requires a negligible amount of computation.
3. For each root μ found in the previous step, check whether the rank of $R(\mu)$ is smaller than or equal to $\lfloor \frac{k-1}{2} \rfloor$.
4. If the unique decodability assumption is satisfied, exactly one root μ will satisfy the rank condition in the previous step. In this case, we decode to $\text{rowsp}(I \quad S\Delta(\mu)S^{-1})$.
5. Else, none of the roots will. In this case, we have a decoding failure.

Algorithm 2 at page 461 represents a detailed minimum-distance decoding algorithm in pseudocode.

3.3. A VERY EFFICIENT DECODING ALGORITHM FOR THE CASE R_1 NON SINGULAR. In this subsection, we focus on the case where the received word $\mathcal{R} = \text{rowsp}(R_1 \quad R_2) \in \mathfrak{G}_{\mathbb{F}_q}(k, n)$ satisfies $R_1 \in GL_k(\mathbb{F}_q)$. In this case, we simplify the decoding algorithm and make its complexity essentially negligible.

We start by establishing the mathematical background. Under the assumption that the matrix R_1 is invertible, an alternative form of Theorem 32 holds.

Proposition 35. *Let $\mathcal{R} \in \mathfrak{G}_{\mathbb{F}_q}(k, n)$ be a subspace with*

$$\frac{k-1}{2} < \text{rank}(R_2) \leq \text{rank}(R_1) = k.$$

The following are equivalent:

- \mathcal{R} is uniquely decodable.
- There exists a unique $\mu \in \mathbb{F}_{q^k}$ such that

$$\text{rank}(\Delta(\mu) - S^{-1}R_1^{-1}R_2S) = \text{ndrank}(S^{-1}R_1^{-1}R_2S).$$

Proof. By Theorem 32 \mathcal{R} is uniquely decodable if and only if there exists a unique $\mu \in \mathbb{F}_{q^k}$ such that

$$\text{rank}(\Delta(\mu) - S^{-1}R_1^{-1}R_2S) \leq \frac{k-1}{2}.$$

Let $A = S\Delta(\mu)S^{-1}$, then by Corollary 27

$$\text{rank}(A - R_1^{-1}R_2) = \text{ndrank}(\Delta(\mu) - S^{-1}R_1^{-1}R_2S) = \text{ndrank}(S^{-1}R_1^{-1}R_2S). \quad \square$$

Our improved decoding algorithm relies on the following corollary.

Algorithm 2: Minimum-distance decoding algorithm: $n = 2k$

input : $\mathcal{R} = \text{rowsp}(R_1 \ R_2) \in \mathfrak{G}_{\mathbb{F}_q}(\tilde{k}, 2k)$ with $(R_1 \ R_2) \in \mathbb{F}_q^{k \times n}$,
 $P \in GL_k(\mathbb{F}_q)$ the companion matrix of $p \in \mathbb{F}_q[x]$ and
 $S \in GL_k(\mathbb{F}_{q^k})$ its diagonalizing matrix.
output: $\mathcal{C} \in \mathcal{S} \subset \mathfrak{G}_{\mathbb{F}_q}(k, n)$ such that $d(\mathcal{R}, \mathcal{C}) < k$, if such a \mathcal{C} exists.

Let $r_i := \text{rank}(R_i)$ for $i = 1, 2$.

1.

if either $r_1 = k$ and $S^{-1}R_1^{-1}R_2S$ is diagonal or $r_1 = 0$ and $r_2 = k$ **then**
 | **return** $\mathcal{R} \in \mathcal{S}$;
end
2.

if either $r_1 \leq \frac{\tilde{k}-1}{2}$ or $r_2 \leq \frac{\tilde{k}-1}{2}$ **then** go to 3.
else go to 4.
3.

Case $r_1 \leq \frac{\tilde{k}-1}{2}$

 // the case $r_2 \leq \frac{\tilde{k}-1}{2}$ is analogous.
 | **return** $\text{rowsp}(0 \ I)$;
4.

Case $\frac{\tilde{k}-1}{2} < r_2 \leq r_1 \leq \tilde{k}$

 // the case $r_1 \leq r_2$ is analogous.
 | Find $M \in GL_k(\mathbb{F}_{q^k})$ such that $MS^{-1}(R_1 \ R_2)S$ is in row reduced echelon form;
 | $R(x) := MS^{-1}R_1S\Delta(x) - MS^{-1}R_2S$;
 | Let $l_1, \dots, l_{\tilde{k}-r_1} \in \{1, \dots, k\}$ the columns of the pivots of $MS^{-1}R_2S$;
 | Let $I' := (1, \dots, k) \setminus (l_1, \dots, l_{\tilde{k}-r_1}, r_1 + 1, \dots, k)$;
 | Find $J', L' \subset I'$ satisfying Condition (6) and set $s := |J|$;
 | Let $K \subset I' \setminus (J' \cup L')$ with $|K| = \lfloor \frac{\tilde{k}+1}{2} \rfloor - \tilde{k} + r_1 - s$;
 | $\mu_i := \left(\frac{[J' \cup (i), L' \cup (i)]_{R(0)}}{[J', L']_{R(0)}} \right)^{[k-i]}$ for $i \in K$;
 | **if** there exists an $i \in K$ such that $\text{rank}(R(\mu_i)) \leq \frac{\tilde{k}-1}{2}$ **then**
 | **return** $\text{rowsp}(I \ S\Delta(\mu_i)S^{-1})$;
 | **else return** there exists no $\mathcal{C} \in \mathcal{S}$ such that $d(\mathcal{R}, \mathcal{C}) < k$;
 | **end**

Corollary 36. Let $\mathcal{R} = \text{rowsp}(R_1 \ R_2) \in \mathfrak{G}_{\mathbb{F}_q}(k, n)$ be uniquely decodable with $k = \text{rank}(R_1) \geq \text{rank}(R_2) > \frac{k-1}{2}$ and $S \in GL_k(\mathbb{F}_{q^k})$ a matrix diagonalizing P . Let $R(x) := \Delta(x) - S^{-1}R_1^{-1}R_2S$. Then, for any choice of tuples of consecutive indices $J, L \subset (1, \dots, k)$ such that $J \cap L = \emptyset$ and $|J| = |L| = \text{ndrank}(S^{-1}R_1^{-1}R_2S)$ it holds that for any $i \in (1, \dots, k) \setminus (J \cup L)$

$$\text{rank} \left(R \left(\left(\frac{[J \cup (i), L \cup (i)]_{S^{-1}R_1^{-1}R_2S}}{[J, L]_{S^{-1}R_1^{-1}R_2S}} \right)^{[k-i]} \right) \right) \leq \frac{k-1}{2}.$$

Hence the unique $\mu \in \mathbb{F}_{q^k}$ from Proposition 35 is

$$\mu = \left(\frac{[J \cup (i), L \cup (i)]_{S^{-1}R_1^{-1}R_2S}}{[J, L]_{S^{-1}R_1^{-1}R_2S}} \right)^{[k-i]}$$

for any choice of $i \in (1, \dots, k) \setminus (J \cup L)$.

Proof. By Proposition 35, there exists a unique μ for which

$$\text{rank}(R(\mu)) = \text{ndrank}(S^{-1}R_1^{-1}R_2S) \leq \frac{k-1}{2}.$$

Hence it suffices to consider minors of $R(x)$ of size $\text{ndrank}(S^{-1}R_1^{-1}R_2S) + 1$.

By Corollary 27, the minor

$$[J \cup (i), L \cup (i)]_{R(x)} = [J, L]_{S^{-1}R_1^{-1}R_2S} x^{[i]} - [J \cup (i), L \cup (i)]_{S^{-1}R_1^{-1}R_2S}$$

is not identically zero. Hence the root

$$\mu = \left(\frac{[J \cup (i), L \cup (i)]_{S^{-1}R_1^{-1}R_2S}}{[J, L]_{S^{-1}R_1^{-1}R_2S}} \right)^{[k-i]}$$

makes $\text{rank}(R(\mu)) = \text{ndrank}(S^{-1}R_1^{-1}R_2S)$. By Proposition 35, μ yields the unique solution to the decoding problem. \square

Remark 37. Algorithm 3 at page 463 is based on the previous corollary and is more efficient than the decoding algorithm presented in [13], since it does not require the use of the Euclidean Algorithm. More precisely, the previous corollary allows us to find a minor (in fact, many of them) whose roots can be directly computed via an explicit formula. Practically, this makes the decoding complexity negligible.

4. ALGORITHMS COMPLEXITIES

In this section, we compute the complexity of some algorithms that we gave in the previous section.

We start by specifying an algorithm for finding tuples $J', L' \subset I'$ needed in Step 4 of Algorithm 2. The algorithm performs only row operations. The pseudocode is given in Algorithm 4, while correctness is proved in the next lemma.

Lemma 38. Let $M \in \mathbb{F}_q^{k \times k}$ be a non diagonal matrix. Algorithm 4 finds two tuples $J, L \subset (1, \dots, k)$ such that $J, L \neq \emptyset$, $J \cap L = \emptyset$, $[J, L] \neq 0$ and $[J \cup (j), L \cup (l)] = 0$ for any $j, l \in (1, \dots, k) \setminus (J \cup L)$, $j \neq l$.

Proof. We start by setting $K = (1, \dots, k)$. The algorithm eventually terminates since $|K|$ strictly decreases after every cycle of the while loop. Moreover, its complexity is bounded by the complexity of the Gaussian elimination algorithm which computes the row reduced echelon form of a matrix of $\mathbb{F}_q^{n \times n}$ in $\mathcal{O}(\mathbb{F}_q; n^3)$ operations.

We have to prove that the returned tuples $J, L \subset (1, \dots, k)$ satisfy the output conditions. Since M is not diagonal, $J, L \neq \emptyset$. The emptiness of $J \cap L$ follows from the fact that J, L are initialized to \emptyset and each time we modify them, we get $J \cup (j)$ and $L \cup (l)$ where $j \neq l$ and j, l are not elements of $J \cup L$.

In order to continue we have to characterize the matrix N . The matrix changes as soon as we find coordinates $j, l \in I$ with $i \neq j$ for which $n_{jl} \neq 0$. The multiplication PN consists of the following row operations

- the i -th row of PN is the i -th row of N for $i \leq j$, and

Algorithm 3: Minimum-distance decoding algorithm: $n = 2k$, R_1 non-singular

input : $\mathcal{R} = \text{rowsp}(R_1 \ R_2) \in \mathfrak{G}_{\mathbb{F}_q}(k, 2k)$ with either $\text{rank}(R_1) = k$ or $\text{rank}(R_2) = k$
 $P \in GL_k(\mathbb{F}_q)$ the companion matrix of $p \in \mathbb{F}_q[x]$ and
 $S \in GL_k(\mathbb{F}_{q^k})$ its diagonalizing matrix.

output: $\mathcal{C} \in \mathcal{S} \subset \mathfrak{G}_{\mathbb{F}_q}(k, n)$ such that $d(\mathcal{R}, \mathcal{C}) < k$, if such a \mathcal{C} exists.

Let $r_i := \text{rank}(R_i)$ for $i = 1, 2$.

```

1.
  | if either  $r_1 = k$  and  $S^{-1}R_1^{-1}R_2S$  is diagonal or  $r_1 = 0$  and  $r_2 = k$  then
  |   | return  $\mathcal{R} \in \mathcal{S}$ ;
  | end

2.
  | if either  $r_1 \leq \frac{k-1}{2}$  or  $r_2 \leq \frac{k-1}{2}$  then go to 3.
  | else go to 4.

3. Case  $r_1 \leq \frac{k-1}{2}$  // the case  $r_2 \leq \frac{k-1}{2}$  is analogous.
  | return  $\text{rowsp}(0 \ I)$ ;

4. Case  $r_1 = k$  // the case  $r_2 = k$  is analogous.
  |  $R(x) := \Delta(x) - S^{-1}R_1^{-1}R_2S$ ;
  |  $s := \text{rank}((1, \dots, \lfloor \frac{k-1}{2} \rfloor); (k - \lfloor \frac{k-1}{2} \rfloor + 1, \dots, k))_{R(0)}$ ;
  |  $\mu := \frac{[(1, 2, \dots, s+1), (1, k-s, \dots, k)]_{R(0)}}{[(2, \dots, s+1), (k-s, \dots, k)]_{R(0)}}$ ;
  | if  $\text{rank}(R(\mu)) \leq \frac{k-1}{2}$  then
  |   | return  $\text{rowsp}(I \ S\Delta(\mu)S^{-1}) \in \mathcal{S}$ ;
  |   | else return there exists no  $\mathcal{C} \in \mathcal{S}$  such that  $d(\mathcal{R}, \mathcal{C}) < k$ ;
  | end

```

- the i -th row of PN is the i -th row of N minus $\frac{n_{i,l}}{n_{j,l}}$ times the j -th row of N , where $N = (n_{j,l})_{1 \leq j, l \leq k}$ for $i > j$.

It follows that the entries of the l -th column of PN are zero as soon as the row index is bigger than j .

We claim that after each cycle of the while loop it holds that $[J, L]_N \neq 0$. We prove it by induction on the cardinality of J and L . Since the matrix M is not diagonal, the while loop will eventually produce tuples $J = (j)$ and $L = (l)$ with $j \neq l$ such that $[J, L]_M \neq 0$. Now suppose that we have J, L such that $J, L \neq \emptyset$, $J \cap L = \emptyset$ and $[J, L]_N \neq 0$ and there exist, following the algorithm, entries $j, l \in I$ with $j \neq l$ such that $n_{j,l} \neq 0$. From the previous paragraph, the only nonzero entry of the row with index j of $(J \cup (j); L \cup (l))_N$, which by construction is the last one, is $n_{j,l}$, hence

$$[J \cup (j), L \cup (l)]_N = n_{j,l}[J, L]_N \neq 0.$$

In order to conclude that $[J, L]_M \neq 0$, it is enough to notice that the row operations bringing $(J; M)_M$ to $(J; M)_N$ are rank preserving.

The property of maximality of the minor $[J, L]_M$ with respect to containment is a direct consequence of the structure of the algorithm. \square

Algorithm 4: Modified Gaussian elimination

input : $M \in \mathbb{F}_q^{k \times k}$ non diagonal matrix.
output: $J, L \subset (1, \dots, k)$ such that $J, L \neq \emptyset$, $J \cap L = \emptyset$, $[J, L] \neq 0$ and $[J \cup (j), L \cup (l)] = 0$ for any $j \neq l \in (1, \dots, k) \setminus (J \cup L)$.
 $J = L = \emptyset$, $K = (1, \dots, k)$, $j = 1$ and $N = (n_{j,l})_{1 \leq j, l \leq k} = M$;
while $K \neq \emptyset$ **do**
 $t := 0$;
 for $l \in K$ and $l \neq j$ **do**
 if $n_{j,l} \neq 0$ and $t = 0$ **then**
 $J = J \cup (j)$, $L = L \cup (l)$ and $K = K \setminus (j, l)$;
 $P = (p_{j',l'})_{1 \leq j', l' \leq k}$ such that $p_{i,i} = 1$ for any $i \in \{1, \dots, k\}$,
 $p_{i,l} = -\frac{n_{i,l}}{n_{j,l}}$ for any $i \in I$ with $i > j$ and $p_{j',l'} = 0$ otherwise;
 $N = PN$;
 $t = 1$;
 end
 end
 if $t = 0$ **then** $K = K \setminus (j)$;
 $j = \min K$;
end
return J, L ;

For simplicity, in the following comparisons we give the minimum-distance decoding complexity only for the case when the received space $\mathcal{R} \in \mathfrak{G}_{\mathbb{F}_q}(k, n)$. This is an upper bound for the complexity in the general case. The precise complexity for the case when $\mathcal{R} \in \mathfrak{G}_{\mathbb{F}_q}(\tilde{k}, n)$, $\tilde{k} < k$ may be obtained via an easy adaptation of our arguments.

Complexity of the decoding algorithm. Algorithm 2 consists of matrix operations over the extension field $\mathbb{F}_{q^k} \supseteq \mathbb{F}_q$. The most expensive of such operations is the computation of the rank of matrices of size $k \times k$, which can be performed via the Gaussian elimination algorithm. The complexities then are as follows:

- The complexity of step 4. is $\mathcal{O}(\mathbb{F}_{q^k}; k^3)$, which corresponds to the computation of $\text{rank}(R(\mu))$.
- The complexity of step 5. is $\mathcal{O}(\mathbb{F}_{q^k}; k^4)$, which corresponds to the computation of $\text{rank}(R(\mu_i))$ for all $i \in K$, where $|K| \leq \lfloor \frac{k-1}{2} \rfloor$.

The overall complexity of Algorithm 2 is then $\mathcal{O}(\mathbb{F}_{q^k}; k^4)$. This makes the complexity of Algorithm 1 $\mathcal{O}(\mathbb{F}_{q^k}; (n-k)k^3)$. Notice that computing the rank of the matrices R_i has complexity $\mathcal{O}(\mathbb{F}_q; (n-k)k^2)$, which is dominated by $\mathcal{O}(\mathbb{F}_{q^k}; (n-k)k^3)$.

Comparison with other algorithms and conclusions. We compare the complexity of Algorithm 1 with other algorithms present in the literature, specifically with the algorithms discussed in Proposition 18. The complexity of the decoding algorithm contained in [8] is $\mathcal{O}(\mathbb{F}_{q^{n-k}}; n^2)$. In order to compare the two complexity estimates, we use the fact that the complexity of the operations on an extension field $\mathbb{F}_{q^s} \supseteq \mathbb{F}_q$

is $\mathcal{O}(\mathbb{F}_q; s^2)$. This is a crude upper bound, and the complexity may be improved in some cases (see, e.g., [5]). Nevertheless, under this assumption the decoding algorithm from [8] has complexity $\mathcal{O}(\mathbb{F}_q; n^2(n-k)^2)$.

Following similar reasoning, the complexity of the decoding algorithm contained in [15] is $\mathcal{O}(\mathbb{F}_q^{n-k}; k(n-k))$, i.e., $\mathcal{O}(\mathbb{F}_q; k(n-k)^3)$.

We conclude that the minimum-distance decoding algorithm presented in this paper has lower complexity than the algorithms in [8] and [15], whenever $k \ll n$. Since this is the relevant case for the applications, the decoding algorithm that we propose constitutes usually a faster option for decoding spread codes.

5. CONCLUSIONS

In this paper we exhibit a minimum distance decoding algorithm for spread codes which performs better than other known decoding algorithms for RSL codes when the dimension of the codewords is small with respect to the dimension of the ambient space.

The problem of extending our decoding algorithm to the case when the dimension of the received space is bigger than the dimension of the codewords remains open. Another natural question arising from this work is finding a generalization of the decoding algorithm to a list decoding algorithm. Theorem 32 can be easily extended for this purpose. Yet finding a way to solve the list decoding problem which requires neither the computation of a gcd, nor the factorization of a minor is a non trivial task.

ACKNOWLEDGMENTS

The authors are grateful to Heide Gluesing-Luerssen for useful discussions and help in the proof of Theorem 25.

REFERENCES

- [1] R. Ahlswede, N. Cai, S.-Y. R. Li and R. W. Yeung, *Network information flow*, IEEE Trans. Inform. Theory, **46** (2000), 1204–1216.
- [2] T. Etzion and N. Silberstein, *Error-correcting codes in projective spaces via rank-metric codes and Ferrers diagrams*, IEEE Trans. Inform. Theory, **55** (2009), 2909–2919.
- [3] T. Etzion and A. Vardy, *Error-correcting codes in projective space*, in “IEEE International Symposium on Information Theory,” (2008), 871–875.
- [4] È. M. Gabidulin, *Theory of codes with maximum rank distance*, Problemy Peredachi Informatsii, **21** (1985), 3–16.
- [5] E. Gorla, C. Puttmann and J. Shokrollahi, *Explicit formulas for efficient multiplication in $GF(3^{6m})$* , in “Selected Areas in Cryptography: Revised Selected Papers from the 14th International Workshop (SAC 2007) held at University of Ottawa” (eds. C. Adams, A. Miri and M. Wiener), Springer, (2007), 173–183.
- [6] J. W. P. Hirschfeld, “Projective Geometries over Finite Fields,” 2nd edition, The Clarendon Press, Oxford University Press, New York, 1998.
- [7] A. Kohnert and S. Kurz, *Construction of large constant dimension codes with a prescribed minimum distance*, in “MMICS” (eds. J. Calmet, W. Geiselmann and J. Müller-Quade), Springer, (2008), 31–42.
- [8] R. Kötter and F. R. Kschischang, *Coding for errors and erasures in random network coding*, IEEE Trans. Inform. Theory, **54** (2008), 3579–3591.
- [9] S.-Y. R. Li, R. W. Yeung and N. Cai, *Linear network coding*, IEEE Trans. Inform. Theory, **49** (2003), 371–381.
- [10] R. Lidl and H. Niederreiter, “Introduction to Finite Fields and their Applications,” revised edition, Cambridge University Press, Cambridge, 1994.

- [11] P. Loidreau, *A Welch-Berlekamp like algorithm for decoding Gabidulin codes*, in “Coding and Cryptography,” Springer, Berlin, (2006), 36–45.
- [12] H. Mahdaviyar and A. Vardy, *Algebraic list-decoding on the operator channel*, in “Proceedings of 2010 IEEE International Symposium on Information Theory,” (2010), 1193–1197.
- [13] F. Manganiello, E. Gorla and J. Rosenthal, *Spread codes and spread decoding in network coding*, in “Proceedings of 2008 IEEE International Symposium on Information Theory,” Toronto, Canada, (2008), 851–855.
- [14] G. Richter and S. Plass, *Fast decoding of rank-codes with rank errors and column erasures*, in “Proceedings of 2008 IEEE International Symposium on Information Theory,” (2004), page 398.
- [15] D. Silva, F. R. Kschischang and R. Kötter, *A rank-metric approach to error control in random network coding*, IEEE Trans. Inform. Theory, **54** (2008), 3951–3967.
- [16] V. Skachek, *Recursive code construction for random networks*, IEEE Trans. Inform. Theory, **56** (2010), 1378–1382.
- [17] A.-L. Trautmann, F. Manganiello and J. Rosenthal, *Orbit codes – a new concept in the area of network coding*, in “2010 IEEE Information Theory Workshop (ITW),” Dublin, Ireland, (2010), 1–4.
- [18] A.-L. Trautmann and J. Rosenthal, *A complete characterization of irreducible cyclic orbit codes*, in “Proceedings of the Seventh International Workshop on Coding and Cryptography (WCC),” (2011), 219–223.

Received September 2011; revised June 2012.

E-mail address: elisa.gorla@unine.ch

E-mail address: felice.manganiello@utoronto.ca

E-mail address: rosenthal@math.uzh.ch